OBJECTIVE MATHEMATICS

Volume 1

Descriptive Test Series

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CHAPTER-1 : SETS, RELATIONS AND FUNCTIONS

Unit Test-1

1. Let U be the set of all people and M = {Males}, S = {College students}, T = {Teenagers}, W = {People having heights more than five feet}. Express each of the following in the notation of set theory.

(i) College students having heights more than five feet.

(ii) People who are not teenagers and have their heights less than five feet.

(iii) All people who are neither males nor teenagers nor college students.

2. If $aN = \{ax : x \in N\}$, describe the set $3N \cap 7N$.

3. A survey conducted on 600 students of B.A. part I classes of a college gave the following report. "Out of 600 students, 307 took economics, 198 took history, 230 took sociology, 65 took history and economics, 45 took economics and sociology, 31 took sociology and history and 10 took all the three subjects. The report sounded very impressive, but the surveyor was fired. Why?

4. Let A, B, C be subsets of the universal set U. If $n(U) = 692$, $n(B) = 230$, $n(C) = 370$, $n(B \cap C) = 20$, $n(A \cap B' \cap C') = 10$, find $n(A' \cap B' \cap C')$.

5. The report of one survey of 100 students stated that the numbers studying the various languages were : Sanskrit, Hindi and Tamil, 5; Hindi and Sanskrit, 10; Tamil and Sanskrit, 8; Hindi and Tamil, 20; Sanskrit 30; Hindi 23; Tamil 50. The surveyor who prepared the report was fired. Why?

6. The set S and E are defined as given below

$S = \{(x, y) : |x-3| < 1 \text{ and } |y-3| < 1\}$

$E = \{(x, y) : 4x^2 + 9y^2 - 32x - 54y + 109 \le 0\}$

then $S \subset E$

7. If A, B are two sets, prove.

$$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$$

Hence or otherwise prove

$$n(A \cup B) = n(A) + n(B) + n(A \cap B)$$

where $n(A)$ denotes the number of elements in A.

8. A relation R is defined on the set Z of integers as follows: $(x, y) \in R' x^2 + y^2 = 25$

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

9. N is the set of natural numbers. The relation R is defined on $N \times N$ as follows

$$(a,b)\, R\, (c,d)' \; a + d = b + c$$

Prove that R is an equivalence relation.

10. Consider the non-empty set consisting of children in a family. state giving reasons whether each of the following relations is (i) Symmetric (ii) Transitive

(a) x is a brother of y. (b) x likes y

11. Let S be the set of all points in a plane. Let R be a relation on S such that for any two points a and b, aRb iff b is within 1 centimetre from a Check R for reflexivity, symmetry and transitivity.

12. Let R be the set of real numbers.

Statement 1 : $A = \{(x, y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R

Statement 2 : $B = \{(x, y) \in R \times R : x = \alpha \cdot y$, for some rational number $\alpha\}$ is an equivalence relation on R.

13. Suppose f is the collection of all ordered pairs of real numbers and $x = 6$ is the first element of some ordered pair in f. Suppose the vertical line through $x = 6$ intersects the graph of f twice. Is f a function? Why or why not?

14. Is $g(x) = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ a function? If this is described by the formula $g(x) = \alpha x + \beta$, then what values should be assigned to α and β?

15. Is the function $f : N \to N$ (N is set of the natural numbers) defined by $f(n) = 2n + 3$ for all $n \in N$ surjective?

16. Are the following sets of ordered pairs functions ? If so, examine whether the mapping is surjective or injective:

(i) $\{(x, y) : x$ is a person, y is the mother of $x\}$

(ii) $\{(a, b) : a$ is a person, b is an ancestor of $a\}$

17. If R is a set of real numbers and $f : R \to R$ is given by the relation $f(x) = \sin x$, $x \in R$ and mapping $g : R \to R$ by the relation $g(x) = x^2$, $x \in R$, then prove that $fog \neq gof$.

18. Let $f : R \to R$ be defined by $f(x) = \cos(5x+2)$. Is f invertible?

19. Let C be the set of complex numbers. Prove that the mapping $f : C \to R$ given by $f(z) = |z|, z \in C$ is neither one-one nor onto.

20. Let $A = R - (3), B = R - \{1\}$. Let $f : A \to B$ be defined by $f(x) = \dfrac{(x-2)}{(x-3)}$. Is f bijective? Give reasons.

21. Find the domain and range of $f(x) = \dfrac{x^2}{(1+x^2)}$ (x real). Is the function one-to-one?

Hints and Solutions

1. (i) Here we have to write down the set of all those people having both the properties: (1) college. students, (2) people having their height more than five feet. Hence it is the set consisting of common elements of the sets S and W, that is the set $S \cap W$.

(ii) The set of people who are not teenagers is the complement of the set T. that is, it is the set $U - T'$ or T'.

Similarly the set of people having height less than five feet is the complement of W, that is it is the set $U - W$ or W'.

Hence the required set in this case is

$$(U - T) \cap (U - W) \text{ or } T' \cap W'$$

(iii) Arguing as in cases (i) and (ii) we see that the required set in this case is

$$(U - M) \cap (U - T) \cap (U - S) = M' \cap T' \cap S'$$

or using De-Morgan's law, the required set is

$$U - (M \cup T \cup S) \text{ } i.e., \text{ } (M \cup T \cup S)'$$

2. 21 N.

According to the given notation,

$$3N = \{3x : x \in N\} = \{3, 6, 9, 12, \ldots\}$$

and $\quad 7N = \{7x : x \in N\}$

$$= (7, 14, 21, 28, 35, 42, \ldots)$$

Hence $3N \cap 7N = \{21, 42, 63, \ldots\}$

$$= \{21x : x \in N\} = 21N$$

3. Data are inconsistent since

$$n(E \cup S \cup H) \sqrt{n(E) + n(S) + n(H)}$$
$$-n(E \cap S) - n(S \cap H)$$
$$- n(H \cap E) + n(E \cap S \cap H)$$

4. $172 \cdot n(A' \cap B' \cap C') = n\{(B' \cap C') \cap A'\}$

$$= n\{(B' \cap C') - n\{(B' \cap C') \cap A\}$$
$$= n(B \cup C)' - n(A \cap B' \cap C')$$
$$= n(U) - n(B \cup C) - n(A \cap B' \cap C')$$
$$= n(U) - n(B) - n(C) + n(B \cap C) - n(A \cap B' \cap C')$$
$$= 692 - 230 - 370 + 90 - 10$$
$$= 172$$

5. Let S, H and T denote the set of students studying Sanskrit, Hindi and Tamil respectively. Then we are given

$$n(S \cup H \cup T) = 100, n(S) = 30, n(H) = 23,$$
$$n(T) = 50, n(S \cap H) = 10, n(H \cap T) = 20,$$
$$n(T \cap S) = 8, \; n(S \cap H \cap T) = 5.$$

Now $\quad n(S \cup H \cup T) = n(S) + n(H) + n(T)$

$$-n(S \cap H) - n(H \cap T) - n(T \cap S)$$
$$+n(S \cap H \cap T)$$
$$= S_1 - S_2 + S_3$$
$$= (30 + 23 + 50) - (10 + 20 + 8) + 5 = 70.$$

But we are given $n(S \cup H \cup T) = 100$.

Hence the data is inconsistent. That is why the surveyor was fired.

6. We first observe that $|x - 3| < 1$

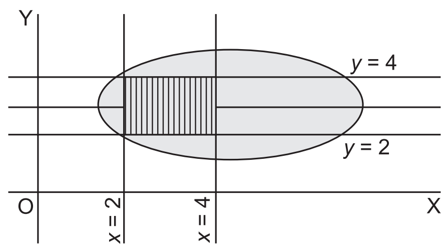
$\Rightarrow \quad -1 < x - 3 < 1$

$\Rightarrow \quad 2 < x < 4$

Similarly $|y - 3| < 1 \to 2 < y < 4$.

Thus S consists of all points inside the square bounded by the lines $x = 2$, $x = 4$, $y = 2$ and $y = 4$.

This square region is shown in the figure by vertical lines.



Again $4x^2 + 9y^2 - 32x - 54y + 109$

$$= 4(x^2 - 8x + 16) + 9(y^2 - 6y + 9) - 36$$
$$= 4(x-4)^2 + 9(y-3)^2 - 36$$

Hence, $4x^2 + 9y^2 - 32x - 54y + 109 \leq 0$

$\Rightarrow \quad 4(x-4)^2 + 9(y-3)^2 - 36 \leq 0$

$$\Rightarrow \quad \frac{(x-4)^2}{9} + \frac{(y-3)^2}{4} \leq 1$$

Thus the set E consists of all points within and on the ellipse whose centre is (4, 3) and semi major and minor axes are 3 and 2 respectively. This region is shown by dots in the diagram. We now show that $S \subset E$.

Let (a, b) be any arbitrary element of S. Then by definition of S, we have

$$2 < a < 4 \text{ and } 2 < b < 4$$

Now $\quad 2 < a < 4 \Rightarrow 2 - 4 < a - 4 < 4 - 4$

$\Rightarrow \quad -2 < a - 4 < 0 \Rightarrow -\dfrac{2}{3} < \dfrac{a-4}{3} < 0$

$\Rightarrow \quad \dfrac{(a-4)^2}{9} < \dfrac{4}{9}$ ...(1)

and $\quad 2 < b < 4$

$\Rightarrow \quad 2 - 3 < b - 3 < 4 - 3$

$\Rightarrow \quad -1 < b - 3 < 1$

$\Rightarrow \quad -\dfrac{1}{2} < \dfrac{b-3}{2} < \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{(b-3)^2}{4} < \dfrac{1}{4}$ ...(2)

From (1) and (2), we have

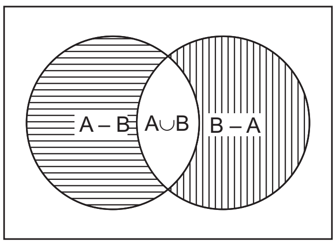
$$\frac{(a-4)^2}{9} + \frac{(b-3)^2}{4} < \frac{4}{9} + \frac{1}{4} = \frac{25}{36} < 1.$$

$\therefore \quad (a, b) \in E$

$\therefore \quad (a, b) \in S \Rightarrow (a, b) \in E$

$\therefore S \subset E$

7.



From the figure, it is clear that

$n(A) = n(A - B) + n(A \cap B)$ ...(1)

or $\quad n(A - B) = n(A) - n(A \cap B)$ ...(2)

Also $\quad n(B - A) = n(B) - n(A \cap B)$ ...(3)

$n(A \cap B) = n(A \cap B)$

Adding (1), (2) and (3), we get

$n(A - B) + n(B - A) + n(A \cap B)$

$= n(A) + n(B) - n(A \cap B)$ ...(4)

But $A - B$, $B - A$ and $A \cap B$ are clearly disjoint and hence

L.H.S. $= n(A \cup B)$

$= n(A - B) + n(B - A) + n(A \cap B)$

or $\quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$ by (4)

Cor. $n(A \Delta B) = n\{(A - B) \cup (B - A)\}$ by def.

$-n(A - B) + n(B - A) - n\{(A - B) \cap (B - A)\}$

$n(A) - n(A \cap B) + n(B) - n(B \cap A) - 0$

$= n(A) + n(B) - 2n(A \cap B)$

Order of Finite Sets: $n(A)$ means the total number of elements in A. They may belong to B also. It does not mean the number of elements which exclusively belong to A.

We give below certain results which are obvious by the help of Venn diagram. If A, B, C be different sets and U be the universal set, then

(1) $n(A') = n(U) - n(A)$

(2) $n(A \cap B') = n(A - B) = n(A) - n(A \cap B)$

(3) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

or $\quad n(A \cap B) = n(A) + n(B) - n(A \cup B)$

(4) $n(A \cup B \cup C) = n(A) + n(B) + n(C)$

$-\{n(A \cap B) - n(B \cap C)$

$- n(C \cap A)\} + n(A \cap B \cap C)$

i.e., $\quad S_1 - S_2 + S_3$

Proof. $n(A \cup B \cup C) = n(A \cup P)$

where $P = B \cup C$

$= n(A) + n(P) - n(A \cap P)$. Put for P

$= n(A) + n(B \cup C) - n\{A \cap (B \cup C)\}$

$= n(A) + n(B \cup C) - n\{(A \cap B) \cup (A \cap C)\}$

by Dist. Law

$= n(A) + n(B \cup C) - n\{L \cup M\}$

$= n(A) + n(B \cup C) - n\{L \cup M\}$

$- \{n(L) + n(M) - n(L \cap M)\}$

$= S_1 - S_2 + S_3$

Note. If A, B, C are all disjoint, then

$n(A \cup B \cup C) = n(A) + n(B) + n(C)$ only.

This can be extended to any number of disjoint sets.

(5) $n(A' \cap B' \cap C') = n(A \cup B \cup C)'$

$= n(U) - n(A \cup B \cup C)$

$= n(U) - \{S_1 - S_2 + S_3\}$

Above stands for the number of elements which do not belong to any of the sets A, B and C.

(6) $n(A \cap B' \cap C')$. This stands for the number of elements which belong to A only *i.e.* they do not belong to both B and C.

$n(A \cap B' \cap C') = n\{A \cap (B \cup C)'\}$

$= n(A) - n\{A \cap (B \cup C)\}$

$= n(A) - n\{(A \cap B) \cup (A \cap C)\}$

$= n(A) - n(P \cup Q)$

$= n(A) - \{n(P) + n(Q) - n(P \cap Q)\}$

$= n(A) - n(A \cap B) - n(A \cap C)$

$+ n\{(A \cap B) \cap (A \cap C)\}$

$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$

$\therefore n(A \cap B' \cap C') = n(A) - n(A \cap B)$

$= n(A \cap C) + n\{A \cap B \cap C\}$

(7) $n(A \cap B \cap C')$. This stands for number of elements

which belong to both A and B but do not belong to C

$$n(A \cap B \cap C') = n(P \cap C')$$
$$= n(P) - n(P \cap C) \qquad \text{by (2)}$$
$$= n(A \cap B) - n(A \cap B \cap C)$$

8. $y = \pm\sqrt{25 - x^4}$

where both x and y are integers.

Clearly for $x = 0; \pm3, \pm4, \pm5$, y will be integers as $y = \pm5, \pm4, \pm3, 0$

$R = \{(0, \pm5), (\pm3, \pm4), (\pm 4, \pm3), (\pm5, 0)\}$

$R^{-1} = \{(+5, 0), (\pm4, \pm3), (\pm3, \pm4), (0, \pm5)\}$

Each R and R^{-1} consists of $2 + 4 + 4 + 2 = 12$ ordered pairs.

Domain of $R = (0, \pm3, \pm4, \pm5)$ = domain of R^{-1}

9. We have $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$ since $a + b = b + a$.

Hence R is reflexive.

R is symmetric. For we have

$$(a,b)\ R\ (c,d) \Rightarrow a + d = b + c$$
$$\Rightarrow \quad d + a = c + b$$
$$\Rightarrow \quad c + b = d + a \quad \Rightarrow \quad (c, d)\ R\ (a,b)$$

R is transitive. For let

$(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$.

Then by definition of R, we have

$$a + d = b + c \text{ and } c + f = d + e,$$

whence by addition, we get

$$a + d + c + f = b + c + d + e$$

or $\quad a + f = b + e$

Hence, $(a,b)\ R\ (e, f)$

Thus $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$

10. (i) $\neq S$ but T

(ii) $\neq S, \neq T$

(a) (i) The given relation is not symmetric, since if x is the brother of y, then y may be the sister of x. Thus in this case $x\,Ry$ but $y\,(\sim R)x$.

(ii) This relation is however transitive, since if x is the brother of y and y is the brother of z, then surely x is the brother of z.

(b) (i) Here the given relation is not symmetric since if x likes y then it is not necessary that y likes x.

(ii) This relation is not transitive since if x likes y and y likes z, then it is not necessary that x likes z.

11. $R, S \neq T$.

R is reflexive since any point is at distance 0 from itself so that it is within 1 centimetre from itself.

R is symmetric since if a point a is within 1 centimetre from another point b, then b is also within 1 centimetre from a.

R is not transitive. Let a, b, c be three points in a straight line in this order such that distance between a and b is $\frac{1}{2}$ centimetre and distance between b and c is $\frac{3}{4}$ centimetre. Then aRb and bRc. But $a(\sim R)c$ since the distance between a and c is $\left(\frac{1}{2} + \frac{3}{4}\right) = \frac{5}{4}$ centimetres which is greater than 1 centimetre.

12. (c)

Statement-1:

(i) x-x is an integer $\forall\ x \in R \Rightarrow A$ is reflexive

(ii) $y - x \in I \Rightarrow x - y \in I \Rightarrow A$ is symmetric

(iii) $y - x \in I$ and $z - y \in I$

$$\Rightarrow \quad z - y + y - x = z - x \in I$$

$\Rightarrow$ A is transitive.

From (i), (ii) and (iii) A is an equivalence relation.

Statement-2:

(i) $x = a$ when $\alpha = 1 \Rightarrow B$ is reflexive

(ii) For $x = 0$ and $y = a$, we have $0 = \alpha$. (2) for $a = 0$
But $2 = a \,.\, 0$ for no $\alpha \Rightarrow B$ is not symmetric

$\Rightarrow \quad B$ is not an equivalence relation.

13. No.

First observe that the graph of the function f consists of points represented by the ordered pairs of the form $(x, f(x))$. If the vertical line through $x = 6$ is cut by the graph of f twice, then it means that the element 6 of the domain of f has two images. Hence f is not a function as for f to be a function each element of the domain must have unique image.

14. $\alpha = 2, \beta = -1$.

Domain $A = \{1, 2, 3, 4\}$

Range $B = \{1, 3, 5, 7\}$

Every element of domain has a unique image in B and hence g is a function. Now $g(x) = \alpha x + \beta$ but $g(2) = 3, g(3) = 5$.

$\therefore \quad 2 = 2\alpha + \beta$ and $5 = 3\alpha + \beta$

$\therefore \quad \alpha = 2, \beta = -1$

15. No. Range will consist of only odd members. Thus even numbers will have no pre-image.

(i) not injective (one-one) but surjective (onto)

(ii) It is not a function.

(i) Here the given set of ordered pairs is a function since each person has one and only one mother. This function is surjective but not injective, (why?). Two persons may have the same mother. It is surjective (onto) as every mother must have a child in A. If in place of mother we had a woman then it may not be onto because every woman need not be a mother.

(ii) Here the given set of ordered pairs is not a function since a person has many ancestors (*e.g.* father, mother, grandfather, grandmother, great grandfather, great grandmother and so in). So in this case the image f (a) of an element a of the domain is not unique.

16. $(fog)(x) = f(g(x)) = f(x^2) = \sin x^2$

and $\quad (gof)(x) = g(f(x)) = g(\sin x) = \sin^2 x$

Since $\sin x^2 \neq \sin^2 x$ for any $x \in R$, $gof \neq fog$.

17. No. For a function to be invertible it is necessary that it is bijective *i.e.* one-one and onto.

The function given here is neither surjective nor injective as shown below.

Since $-1 \le \cos(5x+2) \le 1$, the range of f

$$= \{y : y \text{ is real}, -1 \le y \le 1\}.$$

which is a proper subset of the co-domain R. Hence f is into so that it is not surjective,

f is many-one since cos $(5x+2)$ has the same value for many values of x. Thus

$$f\left(x + \frac{2}{5}n\pi\right) = \cos\left\{5\left(x + \frac{2}{5}n\pi\right) + 2\right\}$$

$$= \cos\{2n\pi + 5x + 2\} = \cos(5x+2) = f(x).$$

for all $n = 0, \pm1, \pm2, \pm3, \ldots.$

Since f is not bijective, it is not invertible.

18. True. f is many-one into as shown below.

Let $\quad z = \cos\theta + i\sin\theta,\ 0 \le \theta \le 2\pi$

Then $f(z) = |z| = +\sqrt{(\cos^2\theta + \sin^2\theta)} = 1$

Thus all complex numbers $z = \cos\theta + i\sin\theta$ where $0 \le \theta \le 2\pi$ have the same image 1. It follows that f is many-one. Again since the modulus of a complex number is a non-negative number, we see that no negative real number can be the image of a complex number. For example, there is no complex number z such that $f(z) = |z| = -1$

Hence f is into. Therefore the function f is neither one-one nor onto.

19. Yes. We have $x_1, x_2 \in \forall\, f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$

$\Rightarrow\ (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow\ x_1x_2 - 2x_2 - 3x_1 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow\ x_1 = x_2$. Hence one-one

Surjectivity. Let y be any arbitrary element of B and suppose there exists an x such that $f(x) = y$, that is,

$$\frac{(x-2)}{(x-3)} = y \text{ or } \quad x - 2 = xy - 3y.$$

This gives $x = \dfrac{(3y-2)}{(y-1)}$. Since $y\sqrt{1}, x$ is real. Further $x\sqrt{3}$. For if $x = 3$, then $3 = \dfrac{(3y-2)}{(y-1)}$ or $3y - 3 = 3y - 2$, which is false. It follows that $x = \dfrac{(3y-2)}{(y-1)} \in A$ such that $f(x) = y$ and so f is surjective. Thus f has been shown to be bijective.

20. Yes. $f^{-1}(y) = \dfrac{1}{3}(y-4), (y \in R)$

(i) f is one-one. For we have

$x_1, x_2 \in R,\ f(x_1) = f(x_2) \quad \Rightarrow 3x_1 + 4 = 3x_2 + 4$

$\Rightarrow \quad x_1 = x_2 f$ is onto :

If $\ y = f(x) = 3x + 4$, then $\ x = \dfrac{1}{3}(y-4)$, which is also a real number. Thus $f\left\{\dfrac{1}{3}(y-4)\right\} = y$. It is therefore shown that any arbitrary element y in R is the f-image of the element $\dfrac{1}{3}(y-4) \in R$. Hence f is onto. Since f is one-one onto (*i.e.*, bijective), it is invertible.

the inverse mapping $f^{-1} : R \to R$ is defined by

$$f^{-1}(y) = \frac{1}{3}(y-4).\ (y \in R)$$

21. Since for every real $x, 1 + x^2 \neq 0$, therefore $\dfrac{x^2}{(1+x^2)}$ is a real number for all real x. Hence the domain of f is the set R of all real numbers. The range of f consists of all real numbers y such that $f(x) = y$ for real x.

Now $\quad f(x) = y \Rightarrow \dfrac{x^2}{(1+x^2)} = y$

$\Rightarrow \quad x^2 = y + yx^2$

$$\Rightarrow \quad x = \sqrt{\left\{\frac{y}{(1-y)}\right\}} \qquad \ldots(1)$$

Since x is real, we must have $\dfrac{y}{(1-y) \ge 0}$ $(y \neq 1)$ which is satisfied if $0 \le y < 1$.

Hence range of $f = \{y : y$ is real and $0 \le y < 1\}$

It is not one-one because $f(x)$ and $f(-x)$ are same.